ON INTRANSITIVE GRAPH-RESTRICTIVE PERMUTATION GROUPS

PABLO SPIGA AND GABRIEL VERRET

Abstract. Let Γ be a finite connected G-vertex-transitive graph and let v be a vertex of Γ. If the permutation group induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$ is permutation isomorphic to L, then (Γ, G) is said to be *locally-L*. A permutation group L is *graph-restrictive* if there exists a constant $c(L)$ such that, for every locally-L pair (Γ, G) and a vertex v of Γ, the inequality $|G_v| \leq c(L)$ holds. We show that an intransitive group is graph-restrictive if and only if it is semiregular.

1. Introduction

A graph Γ is said to be *G-vertex-transitive* if G is a subgroup of $\mathrm{Aut}(\Gamma)$ acting transitively on the vertex-set of Γ. Let Γ be a finite, connected, simple G-vertex-transitive graph and let v be a vertex of Γ. If the permutation group induced by the action of the vertex-stabiliser G_v on the neighbourhood $\Gamma(v)$ is permutation isomorphic to L, then (Γ, G) is said to be *locally-L*. Note that, up to permutation isomorphism, L does not depend on the choice of v, and, moreover, the degree of L is equal to the valency of Γ. In [6, page 499], the second author introduced the following definition.

Definition 1.1. A permutation group L is *graph-restrictive* if there exists a constant $c(L)$ such that, for every locally-L pair (Γ, G) and for every vertex v of Γ, the inequality $|G_v| \leq c(L)$ holds.

To be precise, Definition 1.1 is a generalisation of the definition from [6], where the group L is assumed to be transitive. The problem of determining which transitive permutation groups are graph-restrictive was also proposed in [6]. A survey of the state of this problem can be found in [3], where it was conjectured ([3, Conjecture 3]) that a transitive permutation group is graph-restrictive if and only if it is semiprimitive. (A permutation group is said to be *semiregular* if each of its point-stabilisers is trivial and *semiprimitive* if each of its normal subgroups is either transitive or semiregular.)

Having removed the requirement of transitivity from the definition of graph-restrictive, it is then natural to try to determine which intransitive permutation groups are graph-restrictive. The main result of this note is a complete solution to this problem (which we did not expect, given the abundance and relative lack of structure of intransitive groups).

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Address correspondence to G. Verret (gabriel.verret@fmf.uni-lj.si).

Theorem 1.2. *An intransitive and graph-restrictive permutation group is semiregular.*

It is easily seen that a semiregular permutation group is graph-restrictive. Indeed, if L is a semiregular permutation group of degree d and (Γ, G) is locally-L, then for every arc vw of Γ the group G_{vw} fixes the neighbourhood $\Gamma(v)$ pointwise. Since Γ is connected, it follows that $G_{vw} = 1$ and hence $|G_v| \leq |\Gamma(v)| = d$ and L is graph-restrictive. Thus Theorem 1.2 provides a characterisation of intransitive graph-restrictive groups.

Corollary 1.3. *An intransitive permutation group is graph-restrictive if and only if it is semiregular.*

Note that an intransitive permutation group is semiregular if and only if it is semiprimitive. In particular, Corollary 1.3 completely settles the intransitive version of [3, Conjecture 3], giving remarkable new evidence towards its veracity.

2. Proof of Theorem 1.2

For the remainder of this paper, let L be a permutation group on a finite set Ω which is neither transitive nor semiregular. We show that L is not graph-restrictive, from which Theorem 1.2 follows.

2.1. **The construction.** Let $\omega_1, \ldots, \omega_k \in \Omega$ be a set of representatives of the orbits of L on Ω. Since L is not transitive, $k \geq 2$ and, since L is not semiregular, we may assume without loss of generality that $L_{\omega_1} \neq 1$. Let $n \geq 2$ be an integer and let b_1 be the automorphism of $L_{\omega_1} \times L_{\omega_1}^n = L_{\omega_1}^{n+1}$ defined by

$$(x_0, x_1, \ldots, x_{n-1}, x_n)^{b_1} = (x_n, x_{n-1}, \ldots, x_1, x_0),$$

for each $(x_0, \ldots, x_n) \in L_{\omega_1}^{n+1}$. Similarly, let b_2 be the automorphism of $L_{\omega_1}^n$ defined by

$$(x_1, x_2, \ldots, x_{n-1}, x_n)^{b_2} = (x_n, x_{n-1}, \ldots, x_2, x_1),$$

for each $(x_1, \ldots, x_n) \in L_{\omega_1}^n$. Clearly, b_1 and b_2 are involutions, that is, $b_1^2 = 1$ and $b_2^2 = 1$. Now, let $\langle b_3 \rangle, \ldots, \langle b_k \rangle$ be cyclic groups of order 2 and consider the following abstract groups:

$$\begin{array}{rcll}
A & := & L \times L_{\omega_1}^n, & \\
B_1 & := & \left(L_{\omega_1} \times L_{\omega_1}^n\right) \rtimes \langle b_1 \rangle, & \\
B_2 & := & L_{\omega_2} \times \left(L_{\omega_1}^n \rtimes \langle b_2 \rangle\right), & \\
B_i & := & L_{\omega_i} \times L_{\omega_1}^n \times \langle b_i \rangle, & \text{for } i \in \{3, \ldots, k\}, \\
C_i & := & L_{\omega_i} \times L_{\omega_1}^n, & \text{for } i \in \{1, \ldots, k\},
\end{array}$$

where $b_1, \ldots, b_k \notin A$. For every $i \in \{1, \ldots, k\}$, there is an obvious embedding of C_i in both A and B_i. Hence, in what follows, we regard C_i as a subgroup of both A and B_i. Note that, for each $i \in \{1, \ldots, k\}$, we have $A \cap B_i = C_i$, $|B_i : C_i| = 2$ and $|A : C_i| = |L : L_{\omega_i}|$.

Lemma 2.1. *The core of $C_1 \cap \cdots \cap C_k$ in A is $1 \times L_{\omega_1}^n$.*

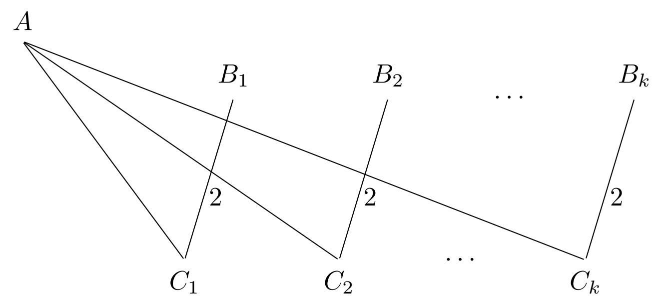

FIGURE 1.

Proof. Let K be the core of $C_1 \cap \cdots \cap C_k$ in A. Then

$$K = \bigcap_{a \in A} (C_1 \cap \cdots \cap C_k)^a = \bigcap_{a \in A} \left((L_{\omega_1} \cap \cdots \cap L_{\omega_k}) \times L_{\omega_1}^n \right)^a .$$

Recall that L is a permutation group on Ω and that $\omega_1, \ldots, \omega_k$ are representatives of the orbits of L on Ω. We thus obtain that $L_{\omega_1} \cap \ldots \cap L_{\omega_k}$ is core-free in L and hence $K = 1 \times L_{\omega_1}^n$. $\square$

Let T be the group given by generators and relators

$$T := \langle A, B_1, \ldots, B_k \mid \mathcal{R} \rangle,$$

where $\mathcal{R}$ consists only of the relations in $A, B_1, \ldots, B_k$ together with the identification of C_i in A and B_i, for every $i \in \{1, \ldots, k\}$. We will obtain some basic properties of T which can be deduced from any textbook on "groups acting on graphs", such as [1, 2, 5].

We have adopted the notation and terminology of [1] and will follow closely [1, I.4]. Using this terminology, the group T is exactly the fundamental group of the graph of groups Y shown in Figure 2. The vertices of Y are A, $B_1, \ldots, B_k$ and, for each $i \in \{1, \ldots, k\}$, there is a (directed) edge C_i from A to B_i.

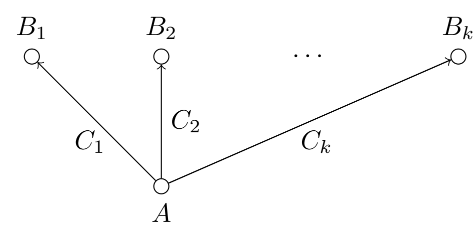

FIGURE 2.

It follows from [1, I.4.6] that the images of $A, B_1, \ldots, B_k, C_1, \ldots, C_k$ in T are isomorphic to $A, B_1, \ldots, B_k, C_1, \ldots, C_k$, respectively. This allows us to identify $A, B_1, \ldots, B_k, C_1, \ldots, C_k$ with their isomorphic images in T in what follows. In particular, for each $i \in \{1, \ldots, k\}$ we still have the equalities $A \cap B_i = C_i$, $|B_i : C_i| = 2$ and $|A : C_i| = |L : L_{\omega_i}|$ in T. Let $\mathcal{T}$ be the graph with vertex-set

$$V\mathcal{T} = T/A \ \sqcup \ T/B_1 \ \sqcup \ \cdots \ \sqcup \ T/B_k,$$

(where $\sqcup$ denotes the disjoint union) and edge-set

$$E\mathcal{T} = \{\{Ax, B_i x\} \mid x \in T, i \in \{1\ldots,k\}\}.$$

2.2. **Results about the group T and the graph $\mathcal{T}$.** Clearly, the action of T by right multiplication on $V\mathcal{T}$ induces a group of automorphisms of $\mathcal{T}$. Under this action, the group T has exactly $k+1$ orbits on $V\mathcal{T}$, namely T/A, $T/B_1,\ldots,T/B_k$, and k orbits on $E\mathcal{T}$ with representatives $\{A,B_1\},\ldots,\{A,B_k\}$. This induces a $(k+1)$-partition of the graph $\mathcal{T}$.

Observe that the set of neighbours of A in T/B_i is $\{B_i a \mid a \in A\}$. As $|A:(A\cap B_i)| = |A:C_i| = |L:L_{w_i}|$, we see that A has $|L:L_{w_i}|$ neighbours in T/B_i. It follows that A has valency $\sum_{i=1}^{k}|L:L_{\omega_i}| = |\Omega|$. A symmetric argument, with the roles of A and B_i reversed, shows that B_i has valency $|B_i:C_i| = 2$. In particular, $\mathcal{T}$ is a $(2,|\Omega|)$-regular graph.

Lemma 2.2. *The stabiliser of the vertex A in T is the subgroup A and the kernel of the action on the neighbourhood of A is $1\times L_{\omega_1}^n$.*

Proof. The definition of $\mathcal{T}$ immediately gives that A is the stabiliser in T of the vertex A. Moreover, the neighbourhood of A is $\mathcal{T}(A) = \{B_i a \mid i \in \{1,\ldots,k\}, a \in A\}$. Let K be the kernel of the action of A on $\mathcal{T}(A)$ and let $x \in K$. Clearly, $B_i a x = B_i a$ if and only if $axa^{-1} \in B_i$, that is, $axa^{-1} \in A\cap B_i = C_i$. It follows by Lemma 2.1 that $K = 1\times L_{\omega_1}^n$. $\square$

One of the most important and fundamental properties of $\mathcal{T}$ is that it is a tree (see [1, I.4.4]). We now deduce some consequences from this pivotal result.

Lemma 2.3. *For each $i \in \{1,\ldots,k\}$, we have $A\cap A^{b_i} = C_i$.*

Proof. We argue by contradiction and assume that $A\cap A^{b_i} \neq C_i$ for some $i \in \{1,\ldots,k\}$. As $|B_i:C_i| = 2$, we see that B_i normalises C_i and hence $C_i < A\cap A^{b_i}$. In particular, there exist $a, a' \in A\setminus C_i$ with $a' = a^{b_i} = b_i^{-1}ab_i$. It follows that A, B_i, Ab_i and $B_i a b_i$ are distinct vertices of $\mathcal{T}$. Now, the definition of $\mathcal{T}$ shows that $(A, B_i, Ab_i, B_i a b_i, Ab_i^{-1}ab_i = A)$ is a cycle of length 4 in $\mathcal{T}$ (see Figure 3). This contradicts the fact that $\mathcal{T}$ is a tree and concludes the proof. $\square$

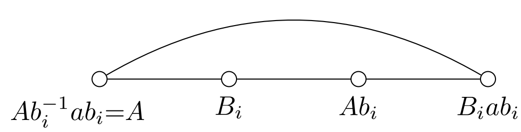

FIGURE 3.

Lemma 2.4. *The subgroup A is core-free in T. In particular, the group T acts faithfully on T/A.*

Proof. Let N be the core of A in T. From Lemma 2.3, we obtain $N \leq C_1\cap\cdots\cap C_k$, and it follows from Lemma 2.1 that $N \leq 1\times L_{\omega_1}^n$. By construction, the group $\langle b_1,b_2\rangle$ induces a transitive permutation group on the $n+1$ coordinates of $L_{\omega_1}\times L_{\omega_1}^n$. As the first coordinate of the elements of N is 1 and as N is invariant under $\langle b_1,b_2\rangle$, we see that every coordinate of N must be equal to 1, that is, $N=1$. The lemma now follows. $\square$

As every vertex of $\mathcal{T}$ not in T/A has valency 2, we see that $\mathcal{T}$ is the subdivision graph of a tree $\mathcal{T}_0$ with vertex set T/A and valency $|\Omega|$. Clearly, T acts transitively and, in view of Lemma 2.4, faithfully on the vertices of $\mathcal{T}_0$. The tree $\mathcal{T}_0$ and the group T are our main ingredients for the proof of Theorem 1.2. (The auxiliary graph $\mathcal{T}$ was introduced mainly to make it more convenient to apply the results from [1].)

Lemma 2.5. *The stabiliser in T of the vertex A of $\mathcal{T}_0$ is the subgroup A and the action induced by A on its neighbourhood is permutation isomorphic to the action of L on Ω.*

Proof. Let $\pi : A \to L$ be the natural projection onto the first coordinate. In other words, if $a = (a_0, a_1, \dots, a_n) \in A$, with $a_0 \in L$ and with $a_1, \dots, a_n \in L_{\omega_1}$, then $\pi(a) = a_0$. Clearly, the kernel of π is $1 \times L^n_{\omega_1}$, which by Lemma 2.2 is also the kernel of the action of A on the neighbourhood of the vertex A. Denote by $\mathcal{T}_0(A)$ the neighbourhood of A in $\mathcal{T}_0$. The definitions of $\mathcal{T}$ and $\mathcal{T}_0$ yield $\mathcal{T}_0(A) = \{Ab_ia \mid i \in \{1, \dots, k\}, a \in A\}$. Let $\varphi : \mathcal{T}_0(A) \to \Omega$ be the mapping $\varphi : Ab_ia \mapsto \omega_i^{\pi(a)}$. We show that φ is well-defined and injective.

Indeed, $Ab_ia = Ab_ia'$ for some $a, a' \in A$ if and only if $Ab_ia(a')^{-1}b_i^{-1} = A$, that is, $a(a')^{-1} \in A \cap A^{b_i}$. By Lemma 2.3, $A \cap A^{b_i} = C_i$. Clearly, $a(a')^{-1} \in C_i$ if and only if $\pi(a(a')^{-1}) \in L_{\omega_i}$, that is, $\omega_i^{\pi(a)} = \omega_i^{\pi(a')}$. This shows that φ is well-defined and that it is a injective.

Clearly, φ is surjective and hence it is a bijection. For every $a, x \in A$ and for every $i \in \{1, \dots, k\}$, we have $\varphi((Ab_ia)x) = (\varphi(Ab_ia))^{\pi(x)}$. As φ is a bijection, this shows that the action of A on $\mathcal{T}_0(A)$ is permutation isomorphic to the action of L on Ω. □

Recall that a group X is said to be *residually finite* if there exists a family $\{X_m\}_{m\in\mathbb{N}}$ of normal subgroups of finite index in X with $\bigcap_{m\in\mathbb{N}} X_m = 1$.

Lemma 2.6. *The group T is residually finite.*

Proof. As the groups A, $B_1, \dots, B_k$ are finite, it follows from [1, I.4.7] that there exists a finite group F and a group homomorphism $\pi : T \to F$ with $\operatorname{Ker}\pi \cap A = 1$ and $\operatorname{Ker}\pi \cap B_i = 1$ for each $i \in \{1, \dots, k\}$. Write $K = \operatorname{Ker}\pi$. Since F is finite, we have $|T : K| < \infty$.

Since $K \trianglelefteq T$, $K \cap A = 1$ and $K \cap B_i = 1$, it follows that the only element of K fixing a vertex of $\mathcal{T}$ is 1 and hence, by [1, I.5.4], K is a free group. In particular, K is residually finite (see [4, 6.1.9] for example). It follows that there exists a family $\{K_m\}_{m\in\mathbb{N}}$ of normal subgroups of finite index in K with $\bigcap_{m\in\mathbb{N}} K_m = 1$.

Let T_m be the core of K_m in T. As $|T : K_m| = |T : K||K : K_m| < \infty$, we see that $|T : T_m| < \infty$. Moreover, since $T_m \leq K_m$, we have $\bigcap_{m\in\mathbb{N}} T_m = 1$ and the lemma follows. □

2.3. **Proof of Theorem 1.2.** We now recall the definition of a normal quotient of a graph. Let Γ be a G-vertex-transitive graph and let N be a normal subgroup of G. Let v^N denote the N-orbit containing $v \in V\Gamma$. Then the *normal quotient* Γ/N is the graph whose vertices are the N-orbits on $V\Gamma$, with an edge between distinct vertices v^N and w^N if and only if there is an edge $\{v', w'\}$ of Γ for some $v' \in v^N$ and some $w' \in w^N$. Observe that the group G/N acts transitively on the graph Γ/N.

Lemma 2.7. *There exists a locally-L pair (Γ_n, G_n) such that the stabiliser of a vertex of Γ_n in G_n has order $|L||L_{\omega_1}|^n$.*

Proof. By Lemma 2.6, T is residually finite and hence there exists a family $\{T_m\}_{m\in\mathbb{N}}$ of normal subgroups of finite index in T with $\bigcap_{m\in\mathbb{N}} T_m = 1$. Consider the set

$$X = \{a_1 b_i^{-1} a_2 b_j a_3 \mid a_1, a_2, a_3 \in A,\ i, j \in \{1, \ldots, k\}\}.$$

Observe that since A is finite, so is X. In particular, as $\bigcap_{m\in\mathbb{N}} T_m = 1$ and $1 \in X$, there exists $m \in \mathbb{N}$ with $X \cap T_m = 1$. Let $G_n = T/T_m$ and $\Gamma_n = \mathcal{T}_0/T_m$. As $|T : T_m| < \infty$, the group G_n and the graph Γ_n are finite. Note that Γ_n is connected and G_n-vertex-transitive. We first show that Γ_n has valency $|\Omega|$.

We argue by contradiction and suppose that Γ_n has valency less than $|\Omega|$. It follows from the definition of normal quotient that the vertex A of $\mathcal{T}_0$ must have two distinct neighbours in the same T_m-orbit. Recall that the neighbourhood of A in $\mathcal{T}_0$ is $\{Ab_i a \mid i \in \{1, \ldots, k\}, a \in A\}$. In particular, $Ab_i a \neq Ab_j a'$ and $Ab_i an = Ab_j a'$, for some $i, j \in \{1, \ldots, k\}$, $a, a' \in A$ and $n \in T_m$. It follows that $n \in a^{-1} b_i^{-1} A b_j a' \subseteq X$ and hence $n \in X \cap T_m = 1$, which is a contradiction.

Let K be the kernel of the action of G_n on $V\Gamma_n$. Since the valency of Γ_n equals the valency of $\mathcal{T}_0$, we have that Γ_n is a regular cover of $\mathcal{T}_0$. Since Γ_n is connected, it follows that K acts semiregularly on $V\Gamma_n$ and hence $K = T_m$. By Lemma 2.5, $(\mathcal{T}_0, T)$ is locally-L and hence so is (Γ_n, G_n). Finally, the stabiliser of the vertex AT_m of Γ_n is $AT_m/T_m \cong A/(A \cap T_m) \cong A$, which has order $|A| = |L||L_{\omega_1}|^n$. $\square$

Proof of Theorem 1.2. By Lemma 2.7, for every natural integer $n \geq 2$, there exists a locally-L pair (Γ_n, G_n) with $|(G_n)_v| = |L||L_{\omega_1}|^n$, for $v \in V\Gamma_n$. As $|L_{\omega_1}| > 1$, this shows that L is not graph-restrictive. $\square$

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Pablo Spiga, Departimento di Matematica Pura e Applicata,
University of Milano-Bicocca, Via Cozzi 53, 20126 Milano, Italy
E-mail address: `pablo.spiga@unimib.it`

Gabriel Verret, Faculty of Mathematics, Natural Sciences and Info. Tech.,
University of Primorska, Glagoljaška 8, 6000 Koper, Slovenia
E-mail address: `gabriel.verret@fmf.uni-lj.si`